Linear irreversible thermodynamics and coefficient of performance

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Following the recent proposal by Van den Broeck for a heat engine [Phys. Rev. Lett. **95**, 190602 (2005)], we analyze the coefficient of performance of a refrigerator in two working regimes using the tools of linear irreversible thermodynamics. In particular, one of the analyzed regimes gives a coefficient of performance which could be considered as the equivalent to the Curzon-Ahlborn efficiency. Also we consider the relation with the Clausius inequality, and some results for the relevant thermodynamic magnitudes in this formalism are confronted with those obtained using the finite-time thermodynamics framework.

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In a recent Letter, Van den Broeck addressed the issue of the efficiency at maximum power for a heat engine and, in particular, the well-known Curzon-Alhborn efficiency [1]. In his Letter, Van den Broeck proposes a general derivation within the realm of linear irreversible thermodynamics (LIT). This is a salient feature because it opens the possibility of analyzing nonisothermal heat engines using the LIT framework, a field up to now almost limited to isothermal energy converters [2].

The main goal of this work is to extend the proposal of Van den Broeck [1] to refrigeration cycles and to study the coefficient of performance (COP) in two different working regimes. The results for refrigerators in the LIT scheme (together with those obtained in [1] for heat engines) will also be faced with predictions of finite-time thermodynamics (FTT), a formalism widely used in the study and optimization of heat devices [3–5].

We start with the analysis of the heat engine depicted in Fig. 1(a). This device extracts a refrigeration load $\dot{Q}$ from a cold space at temperature T_1 at the cost of an expenditure of power $\dot{W}$. We can write the entropy production $\dot{S}$ as

$$\dot{S} = \frac{\dot{W}+\dot{Q}}{T_0} - \frac{\dot{Q}}{T_1} = \frac{\dot{W}}{T_0} + \left(\frac{1}{T_0} - \frac{1}{T_1}\right)\dot{Q}, \tag{1}$$

where T_0 is the temperature of the hotter thermal bath. The definition of the thermodynamic forces and associated fluxes in LIT is to some extent a matter of convention with the sole condition that $\dot{S}=J_1X_1+J_2X_2 \geqslant 0$. Thus, Eq. (1) suggests considering a driver force $X_1=F/T_0$ associated with the external force F performing work with thermodynamically conjugate variable x and a flux $J_1=\dot{x}$, so that $T_0J_1X_1=F\dot{x}=\dot{W}$. As driven force we choose $X_2=(1/T_0-1/T_1)<0$ with a flux given by the cooling power $J_2=\dot{Q}$. With these definitions the COP ϵ is given by

$$\epsilon = \frac{\dot{Q}}{\dot{W}} = \frac{J_2}{T_0J_1X_1}. \tag{2}$$

Assuming X_2 to be a constant and by means of the usual linear relation [6] between forces and fluxes through the direct and cross coupling coefficients L_{ij} ($J_i=\Sigma_j L_{ij}X_j$) we have

$$\epsilon = \frac{1}{T_0X_2}\frac{J_2X_2}{J_1X_1} = -\frac{T_1}{T_0-T_1}\frac{L_{12}X_1X_2+L_{22}X_2^2}{L_{11}X_1^2+L_{12}X_1X_2}. \tag{3}$$

Let us now consider the regime of maximum COP for fixed X_2. Equation (3) presents a maximum when the driver force is

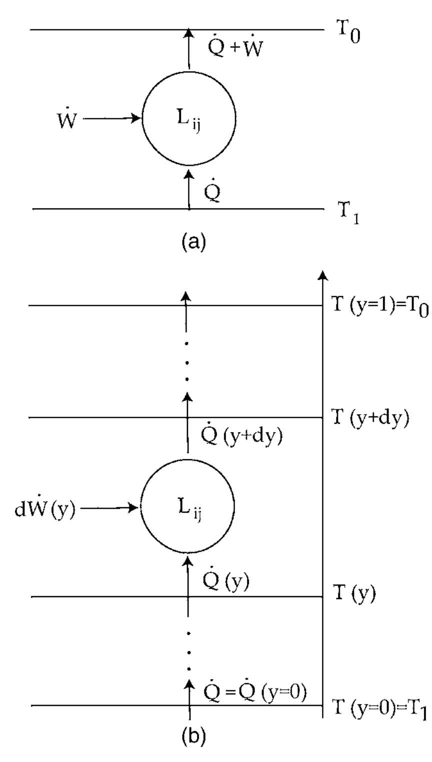

FIG. 1. (a) Generic setup of a refrigerator; (b) cascade construction with a continuum of auxiliary heat baths.

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$$X_1 = -X_2\sqrt{\frac{L_{22}}{L_{11}}}\left(\frac{1+\sqrt{1-q^2}}{q}\right), \quad (4)$$

where q is the usual dimensionless coupling strength $q^2 = L_{12}^2/L_{11}L_{22}$ with condition $-1 \leqslant q \leqslant +1$. The maximum COP is then found to be

$$\epsilon_{max} = \epsilon_C\left(\frac{q^2}{2\sqrt{1-q^2}+(2-q^2)}\right) \equiv \epsilon_C \Phi(q), \quad (5)$$

which ranges between 0 when $|q| \rightarrow 0$ and the Carnot COP $\epsilon_C = \frac{T_1}{T_0 - T_1} \equiv \frac{T_1}{\Delta T}$ when $|q| \rightarrow 1$. According to the LIT formalism Eqs. (4) and (5) are strictly valid when the linear approximation in ΔT holds.

The next step is the maximum COP regime beyond the linear approximation while staying within the LIT framework. Following Van den Broeck [1] we consider the cascade contruction shown in Fig. 1(b): a continuous set of identical refrigerators, each working by means of an external power $d\dot{W}(y)$ between temperatures $T(y)$ and $T(y+dy)$ and under maximum COP conditions with the same coupling strength q. The whole system is sandwiched between the cold $T_1 \equiv T(y=0)$ and hot $T_0 \equiv T(y=1)$ bath temperatures. Thus each refrigerator verifies the conservation of energy $\dot{Q}(y+dy) - \dot{Q}(y) = d\dot{W}(y)$ and works with a COP given by $\dot{Q}(y)/d\dot{W}(y) = \Phi(q)/\left[\frac{d \ln T(y)}{dy}\right]$. Combining the above two relations we have for $d\dot{W}(y)/dy$ the equation

$$\frac{d\dot{W}(y)}{dy} = \frac{1}{\Phi(q)}\frac{d \ln T(y)}{dy}\left(\dot{Q} + \int_0^y \frac{d\dot{W}(y')}{dy'}dy'\right). \quad (6)$$

Differentiating this equation with respect to y and then integrating with the appropriate boundary condition at $y=0$ one obtains the following first-order differential equation:

$$\frac{d\dot{W}(y)}{dy} = \left(\frac{d\dot{W}(y)}{dy}\right)_{y=0}\left(\frac{T(y)}{T(0)}\right)^{1/\Phi(q)}\frac{d \ln T(y)/dy}{[d \ln T(y)/dy]_{y=0}}, \quad (7)$$

which after using Eq. (6) at $y=0$ can be written as

$$\frac{d\dot{W}(y)}{dy} = \frac{\dot{Q}}{T(0)^{1/\Phi(q)}}\frac{d}{dy}[T(y)^{1/\Phi(q)}]. \quad (8)$$

The power input on the whole system is thus given by

$$\dot{W} = \int_0^1 dy \frac{d\dot{W}(y)}{dy} = \dot{Q}\left[\left(\frac{T(1)}{T(0)}\right)^{1/\Phi(q)} - 1\right] \equiv \dot{Q}\left[\left(\frac{T_0}{T_1}\right)^{1/\Phi(q)} - 1\right], \quad (9)$$

and the COP becomes finally

$$\epsilon_{max}(\tau,q) \equiv \frac{\dot{Q}}{\dot{W}} = \frac{T_1^{1/\Phi(q)}}{T_0^{1/\Phi(q)} - T_1^{1/\Phi(q)}} \equiv \frac{\tau^{1/\Phi(q)}}{1-\tau^{1/\Phi(q)}}, \quad (10)$$

where $\tau = T_1/T_0 \leqslant 1$. Equation (10) gives a maximum COP independent of the prescribed temperature profile. It varies between the Carnot value $\epsilon_C = \tau/(1-\tau)$ when $|q|=1$ (perfect coupling) and zero in the absence of any kind of cross coupling, $q=0$. The evolution of the COP with τ is shown in Fig. 2 for some $|q|$ values. In real mechanical chillers typical τ values range between 0.92 and 0.98 and the observed COP's are about between 2 and 4 [7]. These values are matched by Eq. (9) for $|q|$ values in the interval $0.9 \gtrsim |q| \gtrsim 0.7$.

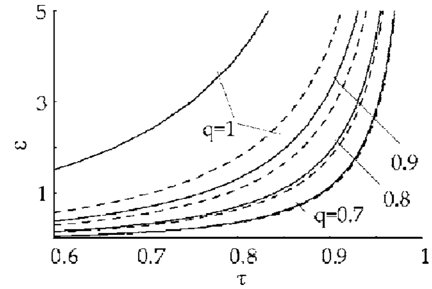

FIG. 2. Behaviors of the maximum COP $\epsilon_{max}(\tau,q)$ (solid line) in Eq. (10) and COP under conditions of maximum J_2X_2, $\epsilon_{max\,J_2X_2}(\tau,q)$, (broken line) in Eq. (11) for the labeled $|q|$-values.

An interesting point in the LIT formalism concerns with its versatility. Since in LIT one is free to choose the fluxes and thermodynamic forces with the sole condition of the positivity of the entropy production ($\dot{S} = J_1X_1 + J_2X_2 \geqslant 0$) it is possible to define different figures of merit or optimization regimes, as happens in FTT [3–5]. Accordingly, both formalisms, although quite different in nature, could be used in a complementary way in order to go deep into the thermodynamical analysis and optimization of real heat devices.

As an illustrative example consider the optimization of the magnitude $J_2X_2 \equiv \dot{Q}(1/T_0 - 1/T_1)$ keeping X_1 fixed. It is easy to show that under linear conditions the COP at maximum J_2X_2 is given by $\epsilon_{max\,J_2X_2} = \epsilon_C\bar{\Phi}(q)/2$ where $\bar{\Phi}(q) = q^2/(2-q^2)$ is exactly the same factor that appears when a heat engine is optimized under maximum power output conditions [1]. The cascade construction is similar to the one presented in Fig. 1(b) and the final result for the COP is

$$\epsilon_{max\,J_2X_2}(\tau,q) = \frac{\tau^{2/\bar{\Phi}(q)}}{1-\tau^{2/\bar{\Phi}(q)}}. \quad (11)$$

If above equation is compared with the result obtained by Van den Broeck in [1] $\eta_{max\,\dot{W}} = 1 - \tau^{\bar{\Phi}(q)/2}$ one should conclude that $\epsilon_{max\,J_2X_2}(\tau,q)$ in Eq. (11) plays the same role in refrigerators that the efficiency at maximum power output in a heat engine. This is not a mere coincidence since the function J_2X_2 for a refrigerator is equivalent to $J_1X_1 \propto \dot{W}$ for a heat engine, both considered as generic energy converters in LIT. In particular, for $q=1$ the function

$$\epsilon_{max\,J_2X_2}(\tau,1) = \frac{\tau^2}{1-\tau^2} = \frac{\tau}{1+\tau}\epsilon_C, \quad (12)$$

could play the role of a Curzon-Ahlborn COP. It should be stressed that in the Carnot-like models (see below) of refrigerators in FTT with two noninstantaneous isotherms and two instantaneous adiabats the power input is not an objective function to be optimized following the Curzon-Ahlborn technique and, as a consequence, a bound like that given by Eq. (12) or Eq. (11) cannot be derived. A number of different optimization criteria for Carnot-like refrigerators can be found, for instance, in Refs. [4,8]. In Fig. 2 we show the behavior of $\epsilon_{max\, J_2X_2}(\tau,q)$ for some $|q|$ values. Note that when the losses are high enough $\epsilon_{max}(\tau,q)$ and $\epsilon_{max\, J_2X_2}(\tau,q)$ become almost identical, i.e., as expected, the refrigerator performance does not distinguish between optimized working regimes.

We stress an important property regarding with the ratio of heat transfers, which emerges from the q independence of the cascade construction and from the particular analytical form of the optimized COPs in Eqs. (10) and (11). If we return to Fig. 1(b) and make $\dot{Q}_H=\dot{Q}(y=0)+\int_{y=0}^{y=1}d\dot{W}(y)$ and $\dot{Q}_L=\dot{Q}(y=0)$ we obtain the equality

$$\left(\frac{\dot{Q}_L}{\dot{Q}_H}\right)_{wr}=\left(\frac{T_1}{T_0}\right)_{wr}^{1/\Phi_{wr}(q)}, \tag{13}$$

where $\Phi_{wr}(q)$ denotes the q function coming from the selected working regime (wr) with boundary conditions $\lim_{|q|\to 1}\Phi_{wr}(q)\leq 1$ and $\lim_{q\to 0}\Phi_{wr}(q)=0$ and where $\dot{Q}_L$ ($\dot{Q}_H$) stands for the heat transfer absorbed (delivered) by the cyclic system at temperature T_1 (T_0). The interesting point in Eq. (13) is that it contains specific information about the losses of the refrigerator through the optimized working regime and the coupling strength value q. It can be written in terms of the Clausius inequality as $\tau^{1/\Phi_{wr}(q)}\leq\tau$. Indeed, under maximum COP conditions [$\Phi_{wr}(q)=\Phi(q)$ in Eq. (5)] and in the limit $|q|\to 1$ Eq. (13) becomes the Clausius equality, but under any other different conditions the term $\tau^{1/\Phi_{wr}(q)}$ quantifies the losses of the refrigerator due to the specific irreversibilities involved both in the working regime and in the concrete value of the coupling. For instance, at maximum J_2X_2 conditions we have that $\tau^{2/\bar{\Phi}(q)}\leq\tau$. Similar results hold for a heat engine. In this case we have that $\left(\frac{\dot{Q}_L}{\dot{Q}_H}\right)_{wr}=\left(\frac{T_1}{T_0}\right)_{wr}^{\rho_{wr}(q)}$, where $\rho_{wr}(q)$ denotes the q function specifying the optimum working regime, and the Clausius inequality reads as $\tau^{\rho_{wr}(q)}\geq\tau$. In the maximum efficiency regime, where it is easy to prove that $\rho_{wr}(q)\equiv\rho_{max\,\eta}(q)=\Phi(q)$, and in the limit $|q|\to 1$ the Clausius equality is recovered.

Independently of any concrete optimized working regime like those above analyzed for refrigerators or the one reported by Van den Broeck [1] for the efficiency at maximum power in a heat engine, it is interesting to face the predicted results for the relevant thermodynamic magnitudes in LIT with those coming from the FTT formalism. Although in the context of FTT there are many different treatments of heat devices based on a variety of different constraints (see, for instance, Refs. [3–5] for an overview) we will focus here on the so-called Carnot-type models. They are widely used in FTT because, in spite of their relative analytical simplicity, are able to account for the main irreversibilities that usually arise in real heat devices: finite-rate heat transfer between the working fluid and the external heat sources, internal dissipation of the working fluid, and heat leak between reservoirs.

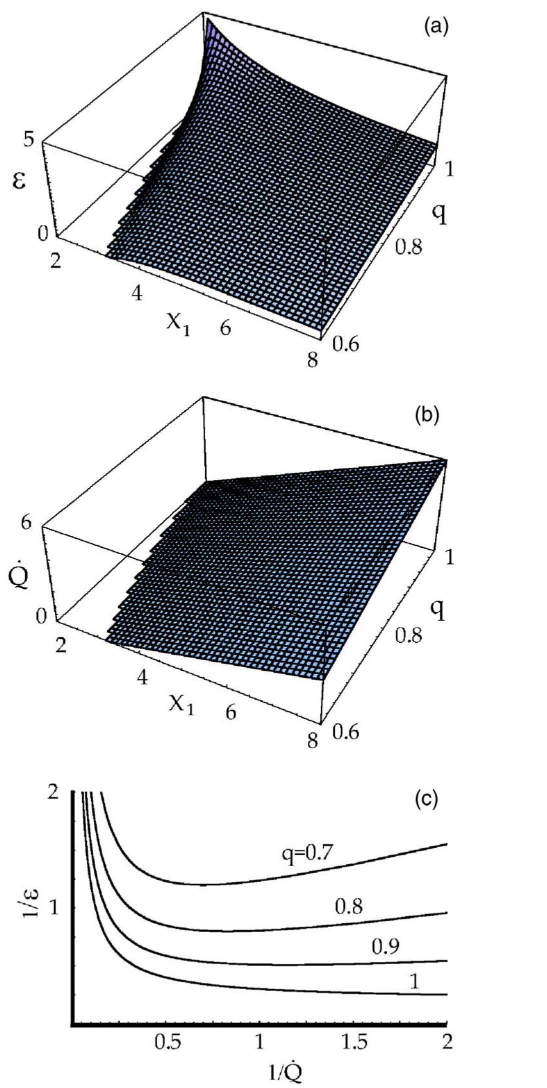

FIG. 3. (Color online) (a) Three-dimensional (3D) plot of the COP ϵ given by Eq. (3) in terms of X_1 and q for $\epsilon_C\equiv T_1/(T_0-T_1)=5$; (b) 3D plot of the cooling power $\dot{Q}\equiv J_2$ in terms of X_1 and q; (c) parametric plot of $1/\epsilon$ versus $1/\dot{Q}$ for the labeled $|q|$ values. In all cases $L_{11}=L_{22}=1$ and $X_2=-2$. All magnitudes are expressed in arbitrary units.

Well-known characteristics of these irreversible Carnot-type models are the following: a loop-shaped power versus efficiency curve for a heat engine [9] and a performance curve of (1/coefficient of performance) against (1/ cooling rate) for a refrigerator where the high cooling rate region is a decreasing function dominated by the external heat transfer losses and the low cooling rate region is an increasing function dominated by internal losses and heat leaks [10]. A particular limit of these models is the so-called endoreversible models where heat leaks are absent. In such a case the power versus efficiency curve of the heat engine becomes an open curve [9] and the (1/coefficient of performance) against (1/cooling rate) curve of the refrigerator becomes a monotonically decreasing function at all cooling rate values [10].

Figure 3(a) shows the behavior of the COP versus our independent variable X_1 at constant X_2 for different $|q|$ values. If the coupling is perfect ($|q|=1$) the COP decreases monotonically with X_1 from its maximum Carnot value to zero, while as q moves to 0 the COP shows a clear maximum value for some X_1 value. The cooling power $\dot{Q}=J_2=L_{12}X_1+L_{22}X_2$ shows [see Fig. 3(b)], a monotonically increasing behavior with X_1 which progressively decreases as $q\rightarrow 0$. The parametric plot $1/\epsilon$ versus $1/\dot{Q}$ is shown in Fig. 3(c). At $|q|=1$ a hyperboliclike behavior is observed, typical of the endoreversible models of a refrigerator in FTT just accounting for external, finite-time heat transfer losses [7,10]. As $|q|$ decreases the contribution of the smaller cooling powers progressively transforms in a monotonic increasing, in agreement with FTT models incorporating additional internal losses [7,10]. It should be mentioned that this behavior, although in qualitative agreement with results of real mechanical chillers, does not reproduce the true evolution of the observed cooling powers under maximum COP conditions [7].

Anyway, the above results suggest that the limit $|q|\rightarrow 1$ in the LIT framework is somehow reminiscent of the endoreversible hypothesis in FTT while results for smaller couplings $|q|<1$ seem to reproduce, qualitatively at least, the predictions of the irreversible models. These results for refrigeration devices also apply for a heat engine. From the results reported by Van den Broeck [1], it is easy to show (see Fig. 4) that the parametric plot efficiency versus power output is an open curve typical of the endoreversible Carnot-like models at $|q|=1$, while for $|q|<1$ the parametric plot becomes loop shaped where maximum efficiency and maximum power are close but noncoincident points, in agreement with results of real heat engines and with predictions of irreversible models in FTT [9].

In summary, we have extended the formalism of Van den Broeck to account for a refrigerator device in the LIT framework and obtained some bounds for the COP in terms of the thermal bath temperatures and a concrete function specifying the selected working regime in such a way that it allows for an easy evaluation of the Clausius inequality. In particular one of the studied working regime gives a COP that could be considered as the equivalent to the Curzon-Alhborn efficiency. The comparison of the reported results and those coming from FTT shows that in LIT a perfect coupling between fluxes and forces ($|q|=1$) quantitatively reproduces an endoreversible FTT model where all irreversibilities are due to external heat transfers, while nonideal couplings ($|q|<1$) reproduce behaviors of the nonendoreversible FTT models with additional internal losses.

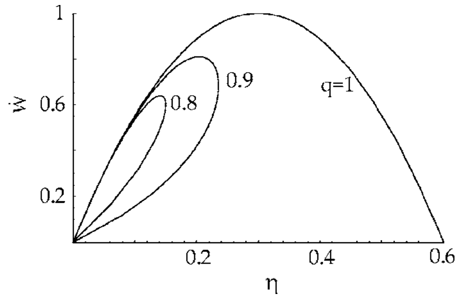

FIG. 4. Parametric plot of the efficiency η versus power output $\dot{W}$ for the motor model by Van den Broeck with the labeled $|q|$ values and a maximum Carnot value of 0.6, $L_{11}=L_{22}=1$, and $X_2=2$. All magnitudes are expressed in arbitrary units.

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